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Gradient estimates for viscosity solutions of singular fully nonlinear elliptic equations

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Abstract

We establish Lipschitz regularity for solutions to a family of non-isotropic fully nonlinear partial differential equations of elliptic type. In general such a regularity is optimal. No sign constraint is imposed on the solution, thus limiting free boundaries may have two-phases. Our estimates are then employed in combination with fine regularizing techniques to prove existence of viscosity solutions to singular nonlinear PDEs.

Keywords: Fully nonlinear elliptic equations; Lipschitz regularity; Singular equations; Solutions with free boundaries

1. Introduction

The regularity theory for uniformly elliptic fully nonlinear operators is, nowadays, fairly well established. For primary results we refer to [4]. However, little is known about optimal regularity for equations with singular terms, for which solutions may exhibit free boundaries. We cite [19], for isotropic equations governed by $F(M) = \inf_\alpha L_\alpha M$, with L_α uniformly elliptic with constant coefficients. For free boundary smoothness results of existing solutions we quote [10,11,17,18].

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The main focus of this paper is to establish Lipschitz a priori estimates for solutions to fully nonlinear equations

$$F\big(D^2u\big) = G\big(x, u, |\nabla u|^2\big) \quad \text{in } \Omega. \tag{1.1}$$

Throughout the paper Ω is a smooth bounded domain in $\mathbb{R}^d$ and $F \in C^1(\mathrm{Sym}(d))$ is uniformly elliptic fully nonlinear operator, i.e., there exist two constants $0 < \lambda \leqslant \Lambda$ such that

$$F(\mathcal{M} + \mathcal{N}) \leqslant F(\mathcal{M}) + \Lambda \|\mathcal{N}^+\| - \lambda \|\mathcal{N}^-\|, \quad \forall \mathcal{M}, \mathcal{N} \in \mathrm{Sym}(d).$$

We shall always normalize it as to $F(0) = 0$ and, unless otherwise stated, we will assume that F satisfies a priori $C^{1,1}$ estimates. Similar result can be derived for nonlinear equations $F(x, u, Du, D^2u) = 0$, with $F \in C^1(\Omega \times \mathbb{R} \times \mathbb{R}^d, \mathrm{Sym}(d))$, uniformly elliptic and satisfying a priori $C^{1,1}$ estimates.

The major concern of our work is to establish gradient estimates that are, in some sense, uniform with respect to analytical properties of the right-hand side G in (1.1). Due to direction dependence of the system, i.e., the non-trivial gradient dependence on G, we say that Eq. (1.1) is non-isotropic. It allows to model systems that depends upon direction. Most of our existence results shall restrict G to the form $\beta(u)\Gamma(|\nabla u|^2)$ with β singular. But more general results can be delivered accordingly.

Primary motivations for our regularity results are in connection with the existence of Lipschitz viscosity solutions to certain free boundary problems. For instance, the following equation

$$F\big(D^2u\big) \approx \frac{\chi_{\{u>0\}}}{u^+} |\nabla u|^m, \tag{1.2}$$

where $m > 0$, $u^+ = \max\{0, u\}$ and $\chi_{\{u>0\}}$ denotes the characteristic function of the set $\{u > 0\}$ appears in the core of modern studies in the theory of free boundary problems. Recently there have been advances in homogeneous but non-isotropic free boundary problems in connection with the theory of flame propagation [6,16]. When Eq. (1.2) is governed by the Laplacian, i.e., $F(\mathcal{M}) = \mathrm{Tr}\, \mathcal{M}$,

$$\Delta u \approx \frac{\chi_{\{u>0\}}}{u^+} |\nabla u|^m, \tag{1.3}$$

Lipschitz regularity for an existing solution can be delivered as in Section 5 of Caffarelli, Jerison and Kenig [6], at least for appropriate exponents m. Their *almost*-monotonicity formula is *ad hoc* for this problem. With crucial impact to the modern theory of free boundary problems, the monotonicity and/or quasi-monotonicity of the Alt–Caffarelli–Friedman functional

$$\Phi(r, u_+, u_-) := \left(\frac{1}{r^2} \int_{B_r} \frac{|\nabla u_+(X)|^2}{|X|^{n-2}}\, dX\right)\left(\frac{1}{r^2} \int_{B_r} \frac{|\nabla u_-(X)|^2}{|X|^{n-2}}\, dX\right) \tag{1.4}$$

seems to be restricted to systems governed by second order elliptic and parabolic operators in divergence form. Therefore new strategies are required in order to study genuine two-phase fully nonlinear singular equation, as in (1.2).

In this present work, we shall establish uniform Lipschitz regularity results, Theorem 1.1 and Corollary 1.2, that yield compactness for a family of solutions of regularized equations to (1.2). This approach provides a systematic existence theory for two-phase fully nonlinear singular PDEs with Lipschitz optimal regularity, for instance, Theorem 1.5. Analogue results can be established for non-homogeneous operators as in Corollary 1.4.

Estimates provided by Corollary 1.3 regard to the case where β is nondecreasing. Such estimates are employed to establish existence of viscosity solutions for two-phase obstacle-type free boundary problem, according Theorem 1.6. Our strategy depends in a decisive way upon uniform control on the first derivatives of approximating solutions. Afterwards, this sequence of solutions is shown to be uniformly $C^{1,\gamma} \cap W^{2,p}$ for any $0 < \gamma < 1$ and $1 < p < \infty$. For problems governed by the Laplacian, we actually show optimal $C^{1,1}$ regularity with the aid of the functional (1.4) of [2], the monotonicity formula of Caffarelli–Kenig [7] and ideas from [15].

It is important to emphasize that in this paper no previously set free boundary condition is imposed for limiting solutions. That is, our limiting free boundary equations are understood as singular PDEs rather than overdetermined problems. We should mention that existence theory for sign-changing solutions to fully nonlinear equations with prescribed jump condition along $\{u = 0\}$ is an outstanding problem, and we do not intend to address that in this paper. On the other hand, when it comes to establishing gradient estimates with the aid of monotonicity and/or almost-monotonicity formulae, it is well established in the theory of free boundary problems that having a prescribed (elliptic) jump condition along the free boundary is crucial for the success of such approach. For details see [8].

We proceed to state our main results. We first establish Lipschitz a priori estimates for solutions to (1.1) based only on the ellipticity constants λ and Λ, the dimension d and on the asymptotic behavior of G.

For a function $\phi : (0, \infty) \to \mathbb{R}$ satisfying $\liminf_{s\to\infty} \phi(s) \geqslant 0$, we define the *asymptotic behavior of* ϕ *passing* 0, $\kappa : (0, 1) \to (0, \infty)$, as

$$\kappa(\varepsilon) := \inf\bigl\{s \bigm| \phi(s) > -\varepsilon\bigr\}.$$

Theorem 1.1. *Let $u \in C^3(\Omega) \cap C(\overline{\Omega})$ be a solution of*

$$\begin{cases} F\bigl(D^2u\bigr) = G\bigl(x, u, |\nabla u|^2\bigr) & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $f \in C^{1,\alpha}(\partial\Omega)$ and $G : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is C^1. Define

$$\sigma\bigl(|p|\bigr) := \inf_{(x,u)} \frac{D_u G(x, u, |p|^2)|p|^2 - |D_x G(x, u, |p|^2)||p|}{G^2(x, u, |p|^2)}, \tag{1.6}$$

and assume

$$S := \liminf_{|p|\to\infty} \sigma\bigl(|p|\bigr) \geqslant 0. \tag{1.7}$$

Then, there exists a constant C, depending only on d, λ, Λ, $\|f\|_{C^{1,\alpha}}$ and the asymptotic behavior of σ passing 0, *such that*

$$\max_{\overline{\Omega}} |\nabla u| \leqslant C.$$

In view of (1.2) we turn our attention to fully nonlinear elliptic equations of the form

$$\begin{cases} F\big(D^2u\big) = \beta(u)\Gamma\big(|\nabla u|^2\big) & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{1.8}$$

Eq. (1.8) by itself accounts for a number of meaningful mathematical problems in biology, chemistry, among others. Furthermore, as we shall see, establishing regularity for Eq. (1.8) under suitable perturbation-invariant hypotheses on β leads to existence and optimal regularity results for certain singular PDEs, which Eq. (1.2) is a particular case. In this direction, we have proven the following result.

Corollary 1.2. *Let u be a $C^3(\Omega) \cap C(\overline{\Omega})$ solution of Eq.* (1.8)*, suppose that $f \in C^{1,\alpha}(\partial\Omega)$, $\beta : \mathbb{R} \to \mathbb{R}$ and $\Gamma : [0, \infty) \to \mathbb{R}$ are C^1. Assume further*

$$L := \inf_u \frac{\beta'(u)}{\beta(u)^2} > -\infty \tag{1.9}$$

and

$$\frac{\Gamma(\tau)}{\tau} \to +\infty \quad \text{as } \tau \to +\infty. \tag{1.10}$$

Then, there exists a constant C, depending only on d, λ, Λ, $\|f\|_{C^{1,\alpha}}$, L and Γ, such that

$$\max_{\overline{\Omega}} |\nabla u| \leqslant C.$$

It is didactically interesting to notice that non-degeneracy condition on β is compatible with Lipschitz renormalization, that is, if for each $\varepsilon > 0$, we define

$$\beta_\varepsilon(s) := \frac{1}{\varepsilon}\beta\left(\frac{s}{\varepsilon}\right)$$

a simple computation shows that β satisfies the non-degeneracy condition (1.9) if and only if β_ε satisfies the same condition with the same constant, in particular, independent of ε.

Despite of interesting applications Corollary 1.2 do have, for instance Theorem 1.5, we should point out that condition (1.9) leaves aside an important example in the theory of free boundary problems. Namely, the non-degeneracy assumption (1.9) is violated when β is a non-negative smooth function supported in [0, 1]. We will come back to this issue at the end of this section.

Under nondecreasing assumption upon β, our regularity result strengthens substantially. Indeed, if β fulfills the right monotonicity assumption, non-degeneracy condition (1.9) is immediately satisfied. In this case, condition on Γ can be relaxed.

Corollary 1.3. *Let $f \in C^{1,\alpha}(\partial\Omega)$ and assume $\beta : \mathbb{R} \to \mathbb{R}$ is C^1, nondecreasing, $|\beta| + |\beta'| > 0$ and $\Gamma : [0, \infty) \to \mathbb{R}$ is C^1 and satisfies*

$$\liminf_{\tau \to \infty} \Gamma(\tau) > 0. \tag{1.11}$$

Then, there exists a constant C, depending only on d, λ, Λ and Γ, but independent of β, such that, for any C^3 solution u to (1.8)*, continuous up to the boundary, we have*

$$\max_{\overline{\Omega}} |\nabla u| \leqslant C.$$

Our approach could also be adapted to deal with non-homogeneous equations.

Corollary 1.4. *Let $F : \Omega \times \mathcal{S}(d) \to \mathbb{R}$ be uniformly elliptic with*

$$\sup_{(x,\mathcal{M})} \left| D_x F(x, \mathcal{M}) \right| \leqslant C_1.$$

Let $f \in C^{1,\alpha}(\partial\Omega)$ and assume either

$$\beta' \geqslant \epsilon_0 > 0 \quad \textit{and} \quad \liminf_{\tau\to\infty} \Gamma(\tau) > 0 \tag{1.12}$$

or

$$0 < \epsilon_0 \leqslant \beta, \quad \beta \textit{ is nondecreasing} \quad \textit{and} \quad \lim_{\tau\to\infty} \frac{\Gamma(\tau)}{\tau^{1/4}} = +\infty. \tag{1.13}$$

Then, there exists a constant C, depending only on d, λ, Λ C_1, ϵ_0 and Γ, but independent of β, such that, for any C^3 solution u of

$$\begin{cases} F\big(x, D^2u\big) = \beta(u)\Gamma\big(|\nabla u|^2\big) & \textit{in } \Omega, \\ u = f & \textit{on } \partial\Omega, \end{cases} \tag{1.14}$$

continuous up to the boundary, we have

$$\max_{\overline{\Omega}} |\nabla u| \leqslant C.$$

Corollary 1.2 can be used to establish existence of Lipschitz viscosity solutions to singular non-isotropic fully nonlinear equations with two-phases. The key point of proof of next theorem is the precise control on the Lipschitz norm of u and how it could deteriorate. This allows us to perform fine regularizing methods to produce Lipschitz viscosity solutions to singular nonlinear PDEs, yet to be accurately stated.

Before presenting next theorem, let us discuss a bit about the notion of viscosity solutions to PDEs with unbounded or singular potentials. We start off with a rigor-free motivation. Consider the one-dimensional Lipschitz function $f(X) = |X|$ in the real line. Clearly f is harmonic in $\{X \neq 0\}$. Indeed, $\Delta f = 2\delta_0$ in the distributional sense. However the key question here is how do we interpret the Laplacian of f at $X = 0$ in the "viscosity sense"? The main difference between distributional and viscosity theories is that the former is an integral theory and the latter is a pointwise theory. Well, by extrapolating the understanding of δ_0 as "function" that vanishes everywhere and equals $+\infty$ at 0, we could say that $\Delta f(0) = +\infty$. In the lights of the viscosity theory, we may obtain an even better justification for the set forth claim. Indeed, given an arbitrary positive number K, $P_K(X) = K|X|^2$ touches f at 0 by below. Indeed, $P(0) = f(0)$ and in $0 < |X| < \frac{1}{K}$, we have $f(X) > P_K(X)$. Thus, "$\Delta f(0) \geqslant K$". After this previous discussion, one might be tempted to give the following definition.

Definition 1. We say a continuous function u satisfies $F(D^2u) = +\infty$ at a point X_0 if its second order superjet at X_0 is empty. Equivalently, u cannot be touched from above by a smooth function at X_0.

Previous definition is a simple generalization to the classical notion of viscosity solution when one allows the right-hand side to become $\pm\infty$. Of course no uniqueness result should be expected with such a definition. The non-uniqueness feature of above definition is perfectly in accordance with the theory of singular PDEs and free boundary problems. It is undertaken by the above notion the fact that u is not $C^{1,1}$ by above. It seems that, in general, this is as much as one can say at a singular point for functions satisfying equations with unbounded potentials. Notice that above notion is also in accordance with the definition of L^p-viscosity solutions introduced in [5].

It turns out that a definition based on an approximative scheme (often called "good solution" in the literature) is more convenient and more robust than the simple generalization suggested above in Definition 1. We will return to this issue at the beginning of Section 3 by providing new definitions of *viscosity solutions*. In the mean time let us state an existence and regularity theorem concerning fully nonlinear singular PDEs.

Theorem 1.5. *Suppose that $f \in C^{1,\alpha}(\partial\Omega)$, $q \geqslant 1$, Γ is a positive $C^{1,\alpha}$ function satisfying* (1.10) *and F is as entitled in the introduction. Then, there exists a Lipschitz viscosity solution in the pointwise topology sense to*

$$\begin{cases} F\big(D^2u\big) = \dfrac{1}{|u|^q}\Gamma\big(|\nabla u|^2\big) & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{1.15}$$

To further elucidate Theorem 1.5, insofar as its regularity result is concerned, we make a parallel to the, by now, well established variational free boundary regularity theory. Consider the functional

$$\mathcal{J}_q(v) := \int_\Omega \frac{1}{2}|\nabla v|^2 + \frac{1}{1-q}\big(v^+\big)^{1-q}\, dX, \tag{1.16}$$

for $q \in [-1, 1)$. The one-phase version of this free boundary problem was studied by Alt and Phillips in [3]. The Euler–Lagrange equation associated to a local minimum is

$$\Delta u = \frac{\chi_{\{u>0\}}}{(u^+)^q} \quad \text{in } \Omega.$$

Notice that $(v^+)^{1-q} \to \chi_{\{v>0\}}$ as $q \nearrow 1$. Thus, the limiting problem as $q \nearrow 1$ is in connection with the work of Alt and Caffarelli [1] on minimizers of

$$\mathcal{J}_0(u) := \int_\Omega \frac{1}{2}|\nabla u|^2 + \chi_{\{u>0\}}\, dX.$$

In that paper, Lipschitz optimal regularity was shown for local minima of $\mathcal{J}_0$. The two-phase case was addressed by Alt, Caffarelli and Friedman [2] where they established Lipschitz regularity for local minima of $\mathcal{J}_0$ with the aid of their revolutionary monotonicity formula.

In order to better appreciate the regularity result offered by Theorem 1.5, we show an informal analysis on a simpler case. Consider equation

$$\Delta u = \frac{1}{(u^+)^q} |\nabla u|^m. \tag{1.17}$$

We will infer optimal regularity by a scaling argument. Define

$$v(X) = \frac{1}{\lambda^\theta} u(\lambda X).$$

An immediate computation reveals that

$$\Delta v = \lambda^{\theta(-q-1+m)+2-m} \left(v^+\right)^{-q} |\nabla v|^m.$$

Thus, if we choose

$$\theta = \frac{m-2}{m-q-1},$$

the rescaled function v satisfies the same equation as u. Therefore optimal regularity for u is expected to be $C^{\frac{m-2}{m-q-1}}$. Requesting $m > 2$ represents the superlinearity condition (1.10) on Γ. For $q = 1$, the borderline for Theorem 1.5, we find $\theta = 1$ which translates into Lipschitz optimal regularity.

It is worthwhile to point out here the regularizing effect of the non-isotropic term $\Gamma(|\nabla u|^2)$ in Eq. (1.15). Such a smoothing feature can be empirically felt by analyzing the computation of the homogeneity of the equation: the key number θ found above. The case $m = 2$ is the borderline for the theory of non-isotropic variational problems as, in principle, just boundedness of distributional solutions should be inferred from the argument drafted above. Nonsingular variational PDEs involving $|\nabla u|^2$ have been extensively studied through the past decades and we do not want to touch such a subject in this article.

Continuing the parallel to the Alt–Phillips theory it is natural to ask whether our approach can furnish viscosity solutions to fully nonlinear PDEs involving potentials of the order u^α for $-1 < \alpha < 0$. The case $\alpha = 0$, i.e. $\Delta u = \chi_{\{u>0\}}$ represents the celebrated obstacle problem. In this direction we have obtained, as a consequence of Corollary 1.3, existence and regularity of solutions to a general two-phase obstacle-type free boundary problems.

Theorem 1.6. *Let $f \in C^{1,\alpha}(\Omega)$ and $\Gamma : [0,\infty) \to \mathbb{R}$ of class $C^{1,\alpha}$ satisfying* (1.11). *Assume $\lambda_- < \lambda_+$. Then there exists a Lipschitz viscosity solution to*

$$\begin{cases} F\left(D^2 u\right) = [\lambda_+ \chi_{\{u>0\}} + \lambda_- \chi_{\{u<0\}}] \Gamma\left(|\nabla u|^2\right) & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{1.18}$$

Furthermore, $u \in C^{1,\gamma}_{\mathrm{loc}}(\Omega) \cap W^{2,p}_{\mathrm{loc}}(\Omega)$, for every $0 < \gamma < 1$ and every $0 < p < \infty$. If $F(\mathcal{M}) = \operatorname{Tr} \mathcal{M}$, then $u \in C^{1,1}(\Omega)$ and this regularity is optimal.

The intermediate cases $-1 < \alpha < 0$ are more delicate and ask for a more robust technique to be dealt with. Of particular interest is existence and optimal regularity for

$$F\big(D^2u\big) = (\alpha+1)\big(u^+\big)^\alpha \chi_{\{u>0\}},$$

with $\alpha \in (-1, 0)$. Here is our main result in this topic, which is in accordance with the version of the problem governed by the Laplacian operator, see [3].

Theorem 1.7. *Let F be as entitled in the introduction and $f \geqslant 0$. For each $-1 < \alpha < 0$, there exists a locally $C^{1,\frac{1-|\alpha|}{1+|\alpha|}}$ viscosity solution in the weak-star topology sense to*

$$\begin{cases} F\big(D^2u\big) = (\alpha+1)\big(u^+\big)^\alpha \chi_{\{u>0\}} & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{1.19}$$

Let us point out that the notion of *viscosity solution in the weak-star topology sense*, see Definition 2 in Section 3, is appropriate for the free boundary problem (1.19). Indeed, one should not expect find a solution in the *viscosity solution in the pointwise topology sense*, Definition 3 in Section 3, since in general $C^{1,\frac{1-|\alpha|}{1+|\alpha|}}$ is the optimal regularity and therefore the Hessian of a solution should blow up at a free boundary point $\partial\{u > 0\}$. For potentials like $|u|^\alpha$, $-1 < \alpha < 0$, once verified non-negativity of solutions, our approach furnishers, with minor modifications, $C^{1,\frac{1-|\alpha|}{1+|\alpha|}}_{\text{loc}}$ *viscosity solution in the pointwise topology sense*. This is in accordance to the optimal regularity, since $|u|^\alpha$ does blow up at free boundary points.

Finally we return to the particularly important free boundary problem earlier mentioned in this introduction: $F(D^2u) = G(x)\zeta_\varepsilon(u)$, where ζ_ε is a suitable approximation to the Dirac mass δ_0. Let us recall the setup of the problem: fix a positive smooth function $\zeta \in C_0^\infty[0, 1]$. For each $\varepsilon > 0$, define

$$\zeta_\varepsilon(t) := \frac{1}{\varepsilon}\zeta\left(\frac{t}{\varepsilon}\right).$$

There are several motivations for the study of equation

$$F\big(D^2u\big) = G(x)\zeta_\varepsilon(u). \tag{1.20}$$

For each ε fixed, Eq. (1.20) models high energy activation problems such as flame propagation, among others. In this setting it is important to establish properties of solutions that are uniform in ε. Eq. (1.20) is also important since is can be seen as a singularly perturbed approximation to certain free boundary problems.

The variational theory for Eq. (1.20) is nowadays well established. When $F(\mathcal{M}) = \operatorname{Tr}\mathcal{M}$, the limiting problem when $\varepsilon \to 0$ was studied by Alt, Caffarelli and Friedman in [2]. As already mentioned, uniform in ε Lipschitz (optimal) regularity for $\Delta u_\varepsilon = \zeta_\varepsilon(u_\varepsilon)$ was proven with the aid of their monotonicity formula combined with "ellipticity" of the free boundary condition: $(u_\nu^+)^2 - (u_\nu^-)^2 = 2$ in some weak sense (see [8] for further details).

As for non-divergence type operators, major difficulties appear when one tries to establish uniform in ε gradient estimate for sign-changing viscosity solutions to (1.20). This is mainly due to the lack of monotonicity formula for equations in non-divergence form. Nevertheless, we have managed to establish uniform in ε Lipschitz (optimal) regularity for such equation, under an extra mild condition. This is the content of next theorem we present.

Theorem 1.8. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, G be a positive C^1 function, F uniform elliptic with $C^{1,1}$ a priori estimates and u_ε be a solution to*

$$F\big(D^2u\big) = G(x)\zeta_\varepsilon(u) \quad \text{in } \Omega,$$

with ζ_ε as above. Assume there exist constants $1 < \lambda_\varepsilon < C$ such that $\{u_\varepsilon = \lambda_\varepsilon \varepsilon\}$ is locally a $C^{1,\alpha}$ surface, with $C^{1,\alpha}$ norm uniformly bounded. Then, given subdomain $\Omega' \Subset \Omega$, there exists a constant K depending on dimension, ellipticity, ζ and G, but independent of ε, such that

$$\max_{x\in\Omega'}\big|\nabla u_\varepsilon(x)\big| \leqslant K.$$

The strategy to show Theorem 1.8 relies on a careful refinement of our arguments previously employed in the proof of Theorem 1.1 and its consequences. Initially, we analyze the "nice" region $\Omega'_\varepsilon := \{y \in \Omega' \mid |u_\varepsilon(y)| > \varepsilon\}$. In this set, u_ε satisfies a homogeneous elliptic PDE. Afterwards we shall control the gradient of u_ε in the "transition area" $\Gamma_\varepsilon := \{y \in \Omega' \mid |u_\varepsilon(y)| \leqslant \varepsilon\}$. As far as we are concerned, up to now, the only successful strategy to handle gradient estimate within the transition area is based on monotonicity formula. This is the main novelty of the present solution.

We finish up this section with few remarks. First we comment on the extra assumption in Theorem 1.8, namely the existence of uniformly smooth level sets. As mentioned before, in $|u_\varepsilon| > \varepsilon$, u_ε satisfies a homogeneous PDE, thus it is smooth. It follows by the classical Sard's Theorem that there is λ_ε, say between 1 and 2 such that $\{u_\varepsilon = \lambda_\varepsilon\varepsilon\}$ is a $C^{1,\alpha}$ surface. Our extra assumption concerns a control on the $C^{1,\alpha}_{\mathrm{loc}}$ norm of these level sets. Such a condition is reasonable since under non-degeneracy condition upon u_ε is it possible to show, with the aid of Caffarelli's free boundary regularity theory generalized to fully nonlinear equations [17,18], that the limiting free boundary is locally a $C^{1,\alpha}$ surface. Non-degeneracy, i.e., linear growth away from zero level set can be shown for Perron solutions to (1.20), see [14]. Furthermore, such assumption can be relaxed. Namely, it suffices to verify that $\{u_\varepsilon > \lambda_\varepsilon\varepsilon\}$ satisfies the exterior sphere condition uniformly in ε, see [12, Chapter 14]. In general, such condition can be obtained via convexity results on level sets of solutions to elliptic PDEs.

Finally, let us make a comment on the C^3 regularity assumption in the statements of Theorems 1.1 and 1.8, and Corollaries 1.2–1.4. It is classical that if the fully nonlinear operator F has a priori $C^{1,1}$ estimates and the right-hand side is of class $C^{1,\alpha}$, then solutions will be locally C^3. The typical example is concave of convex equations (Evans–Krylov Theory) or any fully nonlinear equation in two dimensions. In a recent paper, [13], N. Nadirashvili and S. Vladut constructed a viscosity solution to a fully nonlinear elliptic equation in dimension 24 whose Hessian blows up. In any case we are led to believe that the Lipschitz regularity results presented in this paper hold true even if F does not have a priori $C^{1,1}$ estimates. We leave this as an open problem.

2. Proof of Lipschitz regularity

With a Bernstein-type technique flavor, our proof is based on showing that the maximum point for $|\nabla u|$ can be controlled by a constant.

Proof of Theorem 1.1. Define $v(X) := \frac{1}{2}|\nabla u(X)|^2$. Our goal is to show that v is uniformly bounded. To this end, let $X_0 \in \overline{\Omega}$ be a maximum point of v, that is,

$$v(X_0) = \max_{\overline{\Omega}} v.$$

Recall, that v depends on G, and so does X_0. Thanks to the boundary hypothesis, we may assume X_0 is an interior point; however, no assumption on the distance from X_0 to the fixed boundary, $\partial\Omega$, can be imposed at this moment. A direct computation reveals that

$$D_i v = \sum_k D_k u D_{ki} u. \tag{2.1}$$

Differentiating (2.1) in the j-direction, we obtain

$$D_{ij} v = \sum_k \{D_{jk} u D_{ki} u + u_k D_{ij} u_k\}. \tag{2.2}$$

We will use both notations u_k or $D_k u$ to mean the k-directional derivative. We now differentiate (1.5) to obtain

$$\sum_{ij} F_{ij}\big(D^2 u\big) D_{ij} u_k = \partial_k G + D_u G u_k + 2 D_p G \sum_l u_{lk} u_k. \tag{2.3}$$

By ellipticity, $A_{ij} := F_{ij}(D^2 u(X_0))$ is a positive matrix. Using the fact that X_0 is a maximum point of v and expression (2.2), we obtain

$$\begin{aligned}
0 &\geqslant \sum_{ij} A_{ij} D_{ij} v(X_0) \\
&= \operatorname{Tr}\big(D^2 u(X_0) A_{ij} D^2 u(X_0)\big) + \sum_k u_k \bigg(\sum_{ij} A_{ij} D_{ij} u_k\bigg)(X_0) \\
&\geqslant c_2 \big\| D^2 u(X_0)\big\|^2 + \sum_k u_k \bigg(\sum_{ij} A_{ij} D_{ij} u_k\bigg)(X_0),
\end{aligned} \tag{2.4}$$

where c_2 depends only on dimension and ellipticity. Using ellipticity once more we can compare

$$\big\| D^2 u(X_0)\big\|^2 \geqslant c_3 \big[F\big(D^2 u(X_0)\big)\big]^2. \tag{2.5}$$

From (2.3), we can write

$$\begin{aligned}
&\sum_k u_k \bigg(\sum_{ij} A_{ij} D_{ij} u_k\bigg)(X_0) \\
&\quad = \bigg\{\nabla u \cdot \nabla_x G + D_u G |\nabla u|^2 + 2 D_p G \sum_k u_k \sum_l u_l u_{lk}\bigg\}\big(X_0, u(X_0), \big|\nabla u(X_0)\big|^2\big) \\
&\quad = \big\{\nabla u \cdot \nabla_x G + D_u G |\nabla u|^2\big\}\big(X_0, u(X_0), \big|\nabla u(X_0)\big|^2\big).
\end{aligned} \tag{2.6}$$

The last equality comes from expression (2.1) and the fact that X_0 is a maximum point for v. Combining (2.4)–(2.6) and Eq. (1.8), we obtain at $(X_0, u(X_0), |\nabla u(X_0)|^2)$

$$\nabla u \cdot \nabla_x G + D_u G|\nabla u|^2 \leqslant -c_4 G^2, \tag{2.7}$$

where $c_4 > 0$ is a positive constant that depends only on dimension and ellipticity. Using Cauchy–Schwarz inequality and organizing expression (2.7) we end up with

$$\begin{aligned}\sigma\big(\big|\nabla u(X_0)\big|\big) &\leqslant G^{-2}\big\{D_u G\big|\nabla u(X_0)\big|^2 - |D_x G|\big|\nabla u(X_0)\big|\big\}\big(X_0, u(X_0), \big|\nabla u(X_0)\big|^2\big)\\ &\leqslant -c_4.\end{aligned} \tag{2.8}$$

It follows from the assumption on σ that

$$\max_{\overline{\Omega}} |\nabla u| \leqslant C,$$

for a constant that depends only on dimension, ellipticity and the asymptotic behavior of σ at infinity. The proof of Theorem 1.1 is completed. □

Proof of Corollary 1.2. For Eq. (1.8) we have

$$G\big(x, u, |\nabla u|^2\big) = \beta(u)\Gamma\big(|\nabla u|^2\big).$$

Thus, from the hypotheses on β and Γ, we obtain

$$\begin{aligned}\sigma\big(|p|\big) &:= \inf_{(x,u)} G^{-2}\big(x, u, |p|^2\big)\big\{D_u G\big(x, u, |p|^2\big)|p|^2 - \big|D_x G\big(x, u, |p|^2\big)\big||p|\big\}\\ &= \big(\inf \beta^{-2}\beta'\big)\frac{|p|^2}{\Gamma(|p|^2)}\\ &\geqslant -L\frac{|p|^2}{\Gamma(|p|^2)}\\ &\to 0,\end{aligned}$$

as $|p| \nearrow +\infty$. Furthermore, the asymptotic behavior of σ at infinity is controlled only by L and Γ. That is, given $-\varepsilon < 0$, there exists a constant C depending only on L and Γ, such that

$$\sigma\big(|p|\big) \geqslant -\varepsilon, \quad \forall |p| \geqslant C.$$

Corollary 1.2 follows immediately from Theorem 1.1. □

Next result could be proved in a similar manner of Corollary 1.2. Instead we adopt another strategy.

Proof of Corollary 1.3. For the sake of contradiction, assume that there exists a sequence of non-linearities β_ℓ, satisfying the assumptions of Corollary 1.3, but

$$\frac{1}{\max|\nabla u_\ell|} = o(1) \quad \text{as } \ell \to \infty \tag{2.9}$$

where u_ℓ, is a solution to Eq. (1.8), with β replaced by β_ℓ. We revisit the proof of Theorem 1.1. Initially, from the fixed boundary control on ∇u_ℓ, we know that $\max |\nabla u_\ell|$ must be attained at an interior point X_ℓ. We repeat the proof of Theorem 1.1 until we reach inequality (2.7). Let X_ℓ be a point where $|\nabla u_\ell|$ attains its maximum. At X_ℓ, there holds

$$c_4\beta_\ell^2\big(u_\ell(X_\ell)\big)\Gamma^2\big(\big|\nabla u_\ell(X_\ell)\big|^2\big) + \big|\nabla u_\ell(X_\ell)\big|^2\Gamma\big(\big|\nabla u_\ell(X_\ell)\big|^2\big)\beta_\ell'\big(u_\ell(X_\ell)\big) \leqslant 0, \tag{2.10}$$

where c_4 is a positive constant that depends only on dimension and ellipticity. From the positivity of Γ at infinity and (2.9), we may assume $\Gamma(|\nabla u_\ell|^2) > \delta > 0$. Plugging this into (2.10) we reach a contradiction. □

Proof of Corollary 1.4. We will revisit once more the proof of Theorem 1.1. As in the proof of Corollary 1.3, assume, for sake of contradiction, that there exists a sequence of non-linearities β_ℓ, satisfying the assumptions of Corollary 1.4, but $\max |\nabla u_\ell|$ blows-up. Here u_ℓ denotes a solution to Eq. (1.14), with β replaced by β_ℓ. Let $|\nabla u_\ell(X_\ell)| = \max|\nabla u_\ell|$. As argued before, we may assume X_ℓ is an interior point. Carrying out the very same computations of the proof of Corollary 1.2, but taking into account the x dependence of F, we end up with

$$\begin{aligned} &c_4\beta_\ell^2\big(u_\ell(X_\ell)\big)\Gamma^2\big(\big|\nabla u_\ell(X_\ell)\big|^2\big) + \big|\nabla u_\ell(X_\ell)\big|^2\Gamma\big(\big|\nabla u_\ell(X_\ell)\big|^2\big)\beta_\ell'\big(u_\ell(X_\ell)\big) \\ &\quad - C_1\big|\nabla u_\ell(X_\ell)\big| \leqslant 0, \end{aligned} \tag{2.11}$$

where c_4 is a positive constant that depends only on dimension and ellipticity. Assume (1.12) holds. From the positivity of Γ at infinity we may assume $\Gamma(|\nabla u_\ell|^2) > \delta > 0$. This, together with $\beta_\ell' \geqslant \epsilon_0$, yields

$$0 < \epsilon_0\delta \leqslant \frac{C_1}{|\nabla u_\ell(X_\ell)|} = o(1) \quad \text{as } \ell \to \infty$$

which drives us to a contradiction. If (1.13) holds, from (2.11), we find

$$0 < \frac{c_4\epsilon_0}{C_1} \leqslant \frac{|\nabla u_\ell(X_\ell)|}{\Gamma^2(|\nabla u_\ell(X_\ell)|^2)} = \left[\frac{\sqrt[4]{|\nabla u_\ell(X_\ell)|^2}}{\Gamma(|\nabla u_\ell(X_\ell)|^2)}\right]^2 = o(1) \quad \text{as } \ell \to \infty$$

which again gives us a contradiction. □

3. Existence theory for singular PDEs

We start off this section by introducing new notions of viscosity solutions for problems with measure data and/or singular potentials.

Definition 2. Given a bounded Radon measure μ, we say a continuous function u is a viscosity solution in the weak-star topology sense to $F(D^2u) = \mu$ in Ω if there exist sequences of continuous functions $\{u_j\}_{j\in\mathbb{N}}$ and $\{f_j\}_{j\in\mathbb{N}}$, such that:

(a) u_j converges locally uniformly to u.
(b) f_j converges weak-star to μ, i.e.

$$\int_\Omega f_j(Y)\varphi(Y)\,dY \to \int_\Omega \varphi(Y)\,d\mu(Y), \quad \text{for any } \varphi \in C(\Omega).$$

(c) $F(D^2u_j) = f_j$ in the viscosity sense.

Insofar as the classical viscosity theory is concerned, Definition 2 is a rather weak notion of solution. Namely, if $\mu = g\,dX$, for a continuous function g, then any classical solution to $F(D^2u) = g$ is a weak-star viscosity solution; however the converse may not be true. In light of the variational theory though, one can easily verify that if a continuous function u is a distributional solution to $\Delta u = \mu$, then u is a viscosity solution in the weak-star topology sense to $\Delta u = \mu$ as entitled in previous definition. We also point out that for some non-variational free boundary problems, Definition 2 is the best one can say near free boundary points.

The key observation concerning Definition 2 is that, depending on the nature of the right-hand side of the equation, we ought to choose the appropriate notion of convergence in item (b) that is compatible with the natural space that contains it. Of particular interest to us in this article is pointwise limits, which as we shall see, is a strong notion of convergence, suitable to deal with nonlinear PDEs with singular potentials.

Definition 3. Let Ω be a bounded domain in $\mathbb{R}^d$ and $\beta : \Omega \times \mathbb{R} \to \overline{\mathbb{R}}$ be continuous with respect to x and (possibly) singular with respect to u. We say a continuous function u is a viscosity solution in the pointwise topology sense to $F(D^2u) = \beta(x,u)$ if there exist sequences of continuous functions $\{u_j\}_{j\in\mathbb{N}}$ and $\{\beta_j\}_{j\in\mathbb{N}}$, such that:

(a) u_j converges locally uniformly to u.
(b) $\beta_j(x,\cdot)$ converges pointwisely and locally monotonically to $\beta(x,\cdot)$.
(c) $F(D^2u_j) = \beta_j(x,u_j)$ in the viscosity sense.

Definition of viscosity solution in the pointwise topology sense seems to be suitable for dealing with nonlinear EDPs with unbounded and discontinuous potentials. It is also strong enough to agree with the classical notion of viscosity solutions when the potential is continuous. Next proposition follows by standard arguments in the viscosity theory and therefore its proof will be omitted.

Proposition 3.1. *Assume β is continuous with respect to u, then the classical notion of viscosity solution agrees with the definition of viscosity solution in the pointwise topology sense.*

We now return to existence and regularity issues for a particularly interesting fully nonlinear PDE with singular potential, namely Eq. (1.15). The strategy for proving Theorem 1.5 relies on a regularizing method. We shall approximate the singular term $\beta(u) = |u|^{-q}$ by β_ε, as well as, Eq. (1.15) by ε-perturbed problems (3.1) which have a solution u_ε obtained by a modification of Perron's method. Estimates of Corollary 1.2 allow us to let $\varepsilon \to 0$ and find a *viscosity solution* of (1.15).

Proof of Theorem 1.5. Denote $a := -q \leqslant -1$ and define

$$\beta_\varepsilon(s) := \begin{cases} |s|^a & \text{if } |s| \geqslant \varepsilon, \\ \varphi_\varepsilon(s) & \text{if } -\varepsilon < s < \varepsilon, \end{cases}$$

where φ_ε is defined to be

$$\varphi_\varepsilon(X) := \varepsilon^a \left(\frac{a(a-2)}{8\varepsilon^4} X^4 - \frac{a(a-4)}{4\varepsilon^2} X^2 + 1 - \frac{6a-a^2}{8} \right).$$

One can verify that β_ε, as constructed, is of class C^2. In light of Corollary 1.2, we need to estimate from below $\inf \beta_\varepsilon^{-2}\beta_\varepsilon'$. If $s \leqslant -\varepsilon$, clearly,

$$\inf_{(-\infty,-\epsilon]} \beta_\varepsilon^{-2}\beta_\varepsilon' \geqslant 0,$$

because if $s < -\varepsilon$, β_ε is increasing, thus $\beta_\varepsilon^{-2}\beta_\varepsilon'$ is bounded below by 0. As for the region $s \geqslant \varepsilon$, $\beta_\varepsilon(s) = s^a$, and therefore, for each $K > \varepsilon$, we have

$$\begin{aligned} \inf_{[\varepsilon,K]} \beta_\varepsilon^{-2}\beta_\varepsilon' &= \inf_{[\varepsilon,K]} s^{-2a} a s^{a-1} \\ &\geqslant a K^{-a-1}. \end{aligned}$$

A more careful analysis is needed in the clueing area $[-\varepsilon, \varepsilon]$. Initially we realize that

$$\begin{aligned} \inf_{[-\varepsilon,\varepsilon]} \beta_\varepsilon^{-2}\beta_\varepsilon' = \varepsilon^{-a-1} \inf_{[-1,1]} \Bigg\{ &\left(\frac{a(a-2)}{2} X^3 - \frac{a(a-4)}{2} X \right) \\ &\times \left[\frac{a(a-2)}{8} X^4 - \frac{a(a-4)}{4} X^2 + 1 - \frac{6a-a^2}{8} \right]^{-2} \Bigg\}. \end{aligned}$$

Therefore, we have to show that $\varphi(X) = \frac{a(a-2)}{8} X^4 - \frac{a(a-4)}{4} X^2 + 1 - \frac{6a-a^2}{8}$ is strictly positive in $[-1, 1]$. By evenness of φ, we can restrict our analysis to the interval $[0, 1]$. Define

$$\phi(Y) := a(a-2)Y^2 - 2a(a-4)Y + \left[8 - 6a + a^2\right].$$

Since $\varphi(X) = \frac{1}{8}\phi(X^2)$, we are lead to study ϕ over $[0, 1]$. We claim that ϕ is decreasing on $[0, 1]$. Indeed, recall $a \leqslant -1 < 0$, then

$$\phi'(Y) = 2a(a-2)Y - 2a(a-4) \leqslant -2a(a-4) < 0.$$

Thus, φ is non-increasing in $[0, 1]$, and therefore, $\varphi(X) \geqslant \varphi(1) = 1$. With this in hands, we obtain

$$\inf_{[-\varepsilon,\varepsilon]} \beta_\varepsilon^{-2}\beta_\varepsilon' \geqslant \varepsilon^{-a-1} \inf_{[-1,1]} \left(\frac{a(a-2)}{2} X^3 - \frac{a(a-4)}{2} X \right) > -\infty.$$

The next step is to show that, for each fixed ε, there exists a viscosity solution u_ε to

$$\begin{cases} F(D^2u_\varepsilon) = \beta_\varepsilon(u_\varepsilon)\Gamma(|\nabla u_\varepsilon|^2) & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

We point out that classical Perron's method does not immediately apply to (3.1) since β_ε is not monotone. We will show that this difficulty can be bypassed in this specific case. For that let $u_\star$ and $u^\star$ be functions that agree with f on $\partial\Omega$ and satisfy

$$F(D^2u_\star) = \mathfrak{s}\Gamma(|\nabla u_\star|^2) \quad \text{and} \quad F(D^2u^\star) = 0, \tag{3.2}$$

where $\mathfrak{s} = \mathfrak{s}(a,\varepsilon) := \sup \beta_\varepsilon$. The existence of such functions follows by standard application of Perron's method, see for instance [9, Section 4]. By construction it is clear that $u_\star$ and $u^\star$ are subsolution and supersolution of (3.1) respectively. By comparison principle, $u_\star \leqslant u^\star$ in Ω. Let $\kappa = \kappa(a,\varepsilon) := 3\sup \beta'_\varepsilon$. Consider the fully nonlinear elliptic operator

$$\mathcal{G}(\psi) := F(D^2\psi) - \kappa\psi.$$

It is known that $\mathcal{G}$ satisfies the hypotheses of the comparison principle. We now define a sequence u_k inductively as: $u_0 := u_\star$ and u_k is the unique solution to

$$\begin{cases} \mathcal{G}(u_k) = \beta_\varepsilon(u_{k-1})\Gamma(|\nabla u_{k-1}|^2) - \kappa u_{k-1} & \text{in } \Omega, \\ u_k = f & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

Since $\mathfrak{g}(Z,\xi) := \beta_\varepsilon(Z)\Gamma(\xi) - \kappa Z$ is decreasing on the argument Z, by comparison principle, we have

$$u_\star \leqslant u_1 \leqslant u_2 \leqslant \cdots. \tag{3.4}$$

In the sequel we will show that u_k are all bounded pointwisely by $u^\star$. Initially, we observe that

$$\begin{aligned} \mathcal{G}(u^\star) &= -\kappa u^\star \\ &\leqslant \beta_\varepsilon(u^\star)\Gamma(|\nabla u_\star|^2) - \kappa u^\star \\ &\leqslant \beta_\varepsilon(u_\star)\Gamma(|\nabla u_\star|^2) - \kappa u_\star \\ &= \mathcal{G}(u_1). \end{aligned}$$

Thus, by comparison principle, $u_1 \leqslant u^\star$. Suppose we have shown $u_k \leqslant u^\star$, then, arguing as above,

$$\mathcal{G}(u^\star) \leqslant \beta_\varepsilon(u^\star)\Gamma(|\nabla u_k|^2) - \kappa u^\star \leqslant \beta_\varepsilon(u_k)\Gamma(|\nabla u_k|^2) - \kappa u_k = \mathcal{G}(u_{k+1}),$$

and again the comparison principle implies $u_{k+1} \leqslant u^*$. Finally, defining

$$u_\varepsilon(X) := \lim_{k\nearrow+\infty} u_k(X),$$

we deduce by $C^{1,\alpha}$ elliptic estimates (which, at this stage, depend on ε) that u_k and ∇u_k converge uniformly to u_ε and ∇u_ε, respectively. Standard arguments assure that u_ε is a viscosity solution to (3.1).

We return to our primary analysis. Notice that $F(D^2 u_\varepsilon)$, is uniformly bounded in $|u_\varepsilon| \geqslant 1$. Thus, it follows by the Alexandroff–Bakelman–Pucci maximum principle that

$$|u_\varepsilon| \leqslant \|f\|_{L^\infty} + C \leqslant C.$$

Furthermore, by elliptic regularity, u_ε is locally a C^3 function. We are in the position to invoke Corollary 1.2 and assure that there exists a constant $C = C(n, \lambda, \Lambda, \alpha, \|f\|_{C^{1,\alpha}})$, independent of ε, such that $|\nabla u_\varepsilon(X)| \leqslant C$ in $\overline{\Omega}$.

Therefore, up to a subsequence, u_ε converges uniformly to a Lipschitz function u in Ω. It remains to verify that u is a viscosity solution to the singular PDE (1.15). Let Z be a generic point in $\{X \in \Omega \mid u(X) < 0\}$, say $u(Z) = -\delta < 0$. By continuity, there exists an $r > 0$, such that

$$u(X) < -\frac{\delta}{2}, \quad \forall X \in \overline{B}_r(Z).$$

If $\varepsilon \ll 1$, $u_\varepsilon(X) < -\frac{\delta}{4}$ in $\overline{B}_r(Z)$, thus $\beta_\varepsilon(u_\varepsilon) \equiv |u_\varepsilon|^a$ in $\overline{B}_r(Z)$ and

$$F\big(D^2 u_\varepsilon\big) = |u_\varepsilon|^a \Gamma\big(|\nabla u_\varepsilon|^2\big) \quad \text{in } B_r(Z).$$

By uniform Lipschitz regularity of u_ε, Corollary 1.2, we conclude that there exists a constant M that depends on a, Γ and δ, but independent of ε, such that

$$F\big(D^2 u_\varepsilon\big) \leqslant M.$$

By Caffarelli's $W^{2,p}$ estimate (see [4, Chapter 7]) and Sobolev embedding, we have a control on the $C^{1,\alpha}$ norm of u_ε in $B_{\frac{r}{2}}(X_0)$. In particular, we can assume $|\nabla u_\varepsilon|^2$ converges locally uniformly to $|\nabla u|^2$ in $B_{\frac{r}{4}}(X_0)$. Now by standard arguments we conclude

$$F\big(D^2 u\big) = |u|^a \Gamma\big(|\nabla u|^2\big) \quad \text{in } B_{\frac{\rho}{4}}(X_0) \tag{3.5}$$

in the viscosity sense. Analogously, we prove (3.5) for any point in the set of positivity of u, $\{X \in \Omega \mid u(X) > 0\}$.

It remains to show pointwise limit for any free boundary point $X_0 \in \{u = 0\} \cap \Omega$. Since Γ is positive and $|\nabla u_\varepsilon|$ is uniformly bounded, we know $\Gamma(|\nabla u_\varepsilon|^2) \geqslant \gamma > 0$, uniformly in ε. Thus, we have to verify that for any point $X_0 \in \{u = 0\}$,

$$F\big(D^2 u_\varepsilon(X_0)\big) \to +\infty,$$

as $\varepsilon \to 0$. But this easily follows from the equation u_ε is a solution of, namely, Eq. (3.1) and the uniform bound on ∇u_ε. The proof of Theorem 1.5 is concluded. □

In the next proof we use the estimate from Corollary 1.3 combined with ideas from [15]. We also use the functional (1.4) of [2] and the monotonicity formula [7].

Proof of Theorem 1.6. Let ρ be a smooth function supported in $[0,1]$ and positive in $(0,1)$. Normalize it as to $\int_{\mathbb{R}} \rho = 1$. Define the approximation of β by

$$\beta_\epsilon(s) := \frac{1}{2}(\lambda_+ - \lambda_-)\int_0^{s/\epsilon} \rho(\tau)\,d\tau + \frac{1}{2}(\lambda_- - \lambda_+)\int_0^{-s/\epsilon} \rho(\tau)\,d\tau + \frac{1}{2}(\lambda_+ + \lambda_-) + \epsilon.$$

Notice that

$$\beta_\epsilon(s) \equiv \lambda_- + \epsilon \quad \text{for } s \leqslant -\epsilon,$$
$$\beta_\epsilon(s) \equiv \lambda_+ + \epsilon \quad \text{for } s \geqslant \epsilon.$$

A direct computation yields

$$\beta'_\epsilon(s) = \frac{\lambda_+ + \lambda_-}{2\epsilon}\left[\rho\left(\frac{s}{\varepsilon}\right) + \rho\left(\frac{-s}{\varepsilon}\right)\right].$$

Therefore, $\beta'_\epsilon(s) > 0$ in $(-\epsilon, 0) \cup (0, \epsilon)$. Also, $\beta(0) = (\lambda_+ + \lambda_-) + \epsilon > 0$. Thus, $|\beta_\epsilon| + |\beta'_\epsilon| > 0$ for $\epsilon \ll 1$.

Since β_ϵ has been shown to be nondecreasing, for each ϵ, Perron's method furnishes a viscosity solution u_ϵ to

$$\begin{cases} F\big(D^2 u_\epsilon\big) = \beta_\epsilon(u_\epsilon)\Gamma\big(|\nabla u_\epsilon|^2\big) & \text{in } \Omega, \\ u_\epsilon = f & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

From Corollary 1.3, there exists a constant C_1, independent of ϵ, such that

$$\|\nabla u_\epsilon\|_{L^\infty(\Omega)} \leqslant C_1. \tag{3.7}$$

It now follows from Caffarelli's $W^{2,p}$ estimates that, for any $\Omega' \Subset \Omega$, there exits a constant C_2 depending on Ω', dimension and ellipticity, but independent of ϵ, such that

$$\|u_\epsilon\|_{W^{2,p}(\Omega')} \leqslant C_2.$$

Therefore, up to a subsequence, u_ϵ converges in the $W^{2,p}$ topology to a function u in Ω. Standard arguments show that u is a viscosity solution to (1.18).

We proceed to the proof of optimal regularity when $F(\mathcal{M}) = \operatorname{Tr}\mathcal{M}$. We use an idea of Shahgholian [15]. We are going to show that, for a fixed direction e, $|\nabla D_e u_\epsilon|$ is uniformly bounded, independently of ϵ. As we shall see, Lipschitz estimate (3.7) will play a fundamental role in our proof.

For a fixed direction e, we denote $v^\epsilon(x) := D_e u_\epsilon(x)$. We claim that there exists a uniformly bounded (independent of ϵ) vector field $\vec{\mathbf{F}}_\epsilon$ such that

$$\Delta v^\epsilon_\pm - \vec{\mathbf{F}}_\epsilon \cdot \nabla v^\epsilon_\pm \geqslant 0 \quad \text{in } B_{1/2}, \tag{3.8}$$

in the distributional sense, where v_+^ϵ and v_-^ϵ stand for the positive and negative part of v^ϵ. To verify that, define

$$\Lambda_\pm^\epsilon := \left\{x \in B_{1/2} \;\middle|\; v_\pm^\epsilon(x) > 0\right\}.$$

In Λ_+^ϵ we have

$$\begin{aligned}\Delta v_+^\epsilon &= \beta_\epsilon'(u_\epsilon)v_+^\epsilon + 2\beta_\epsilon(u_\epsilon)\Gamma'\big(|\nabla u_\epsilon|^2\big)\nabla u_\epsilon \cdot \nabla v_+^\epsilon \\ &\geqslant \overrightarrow{\mathbf{F}}_\epsilon \cdot \nabla v^\epsilon,\end{aligned}$$

where

$$\overrightarrow{\mathbf{F}}_\epsilon := 2\beta_\epsilon(u_\epsilon)\Gamma'\big(|\nabla u_\epsilon|^2\big)\nabla u_\epsilon,$$

which is uniformly bounded due to Lipschitz estimate previously proven (3.7). Similar computation holds in Λ_-^ϵ. We conclude that

$$\Delta v_\pm^\epsilon - \overrightarrow{\mathbf{F}}_\epsilon \cdot \nabla v_\pm^\epsilon \geqslant 0 \quad \text{in } \Lambda_\pm^\epsilon$$

and (3.8) follows readily from the fact that if a non-negative function is subharmonic on its positive set, then it is subharmonic in the whole domain.

Our next step is to estimate $|\nabla v^\varepsilon|$ in terms of the Alt–Caffarelli–Friedman functional defined in (1.4). Our analysis will be punctual, say, at 0 and $\epsilon > 0$ will be freezed. We may assume $|\nabla u_\epsilon(0)| > \gamma > 0$. Fix a vector $e = e(\epsilon)$ such that

$$\big\langle e, \nabla u_\epsilon(0)\big\rangle = 0.$$

For such a fixed direction e, we consider v^ϵ as before, that is, $v^\epsilon := D_e \nabla u$ and label $\xi_\epsilon := \nabla v^\epsilon(0)$. Expanding v^ϵ around 0, we have

$$v^\epsilon(x) = \xi_\epsilon \cdot x + o\big(|x|\big).$$

Since our final goal is to estimate $|\xi_\epsilon|$ we may assume $\xi_\epsilon \neq 0$. We now consider the Lipschitz re-scaling of v^ϵ defined on B_1 by

$$v_\delta^\epsilon(x) := \frac{1}{\delta} v^\epsilon(\delta x).$$

One easily verifies that $\nabla(v_\delta^\epsilon)_\pm \to \xi_\epsilon$, pointwisely in Ω as $\delta \to 0$. Next we consider the cone C_ϵ of all vectors in $\mathbb{R}^n$ that makes an angle less or equal to $\frac{\pi}{3}$, with ξ_ϵ, that is $\mathrm{C}_\epsilon := \{x \in \mathbb{R}^n \mid \xi_\epsilon \cdot x \geqslant \frac{1}{2}|\xi_\epsilon||x|\}$. For $r \ll 1$, we know that

$$\mathrm{C}_\epsilon \cap B_r \subset \big\{v^\epsilon > 0\big\} \quad \text{and} \quad -\mathrm{C}_\epsilon \cap B_r \subset \big\{v^\epsilon < 0\big\}.$$

As a consequence of Fatou's Lemma, we have

$$
\begin{aligned}
c_n\left|\nabla v^\epsilon(0)\right|^4 &\leqslant \liminf_{\delta\searrow 0}\int_{B_1}\frac{|\nabla(v^\epsilon_\delta)_+(X)|^2}{|X|^{n-2}}\,dX\int_{B_1}\frac{|\nabla(v^\epsilon_\delta)_-|^2(X)}{|X|^{n-2}}\,dX\\
&\leqslant \liminf_{\delta\searrow 0}\frac{1}{\delta^4}\int_{\mathrm{C}_\epsilon\cap B_\delta}\frac{|\nabla v^\epsilon_+(X)|^2}{|X|^{n-2}}\,dX\int_{-\mathrm{C}_\epsilon\cap B_\delta}\frac{|\nabla v^\epsilon_-|^2(X)}{|X|^{n-2}}\,dX\\
&\leqslant \liminf_{\delta\searrow 0}\Phi\left(\delta, v^\epsilon_+, v^\epsilon_-\right)
\end{aligned}
\tag{3.9}
$$

where Φ is the Alt–Caffarelli–Friedman functional (1.4) defined in (1.4).

Since we have proven that v^ϵ_+ and v^ϵ_- are subsolutions of an elliptic equation with a uniformly bounded drift $\overrightarrow{\mathbf{F}}_\epsilon$, Eq. (3.8), we are in position to invoke the elliptic version of Caffarelli–Kenig parabolic monotonicity formula [7], to conclude that there exists a constant C depending only on dimension and $\|\overrightarrow{\mathbf{F}}_\epsilon\|_{L^\infty(B(3/4))}$, in particular independent of ϵ, such that

$$
\begin{aligned}
\left|\nabla v^\epsilon(0)\right|^4 &\leqslant C\lim_{r\to 0}\Phi\left(r, v^\epsilon_+, v^\epsilon_-, y\right)\\
&\leqslant C\Phi\left(1/2, v^\epsilon_+, v^\epsilon_-, y\right)\\
&\leqslant C\left\|\nabla u^\epsilon\right\|^4_{L^2(B_1)}\\
&\leqslant C,
\end{aligned}
\tag{3.10}
$$

because of (3.7). Recall that in order to derive (3.10) we had to restrict our analysis to vectors e orthogonal to $\nabla u^\epsilon(0)$. To finally conclude we argue as in [15]. If $\nabla u^\epsilon(0)$ we choose a system of coordinates so that $\nabla u^\epsilon(0)$ is parallel to e_1. Applying (3.10) for $e = e_2, e_3, \ldots, e_n$, we find

$$
\left|\partial_{x_i}\partial_{x_j}u^\epsilon\right| \leqslant C, \quad \forall i = 2, 3, \ldots, n \text{ and } \forall j = 1, 2, \ldots, n.
$$

The only estimate missing is $D_{11}u^\epsilon$. For that we recur to the equation and (3.7) to get

$$
\begin{aligned}
\left|D_{11}u^\epsilon\right| &\leqslant \left|\Delta u^\epsilon\right| + \sum_{j=2}^{n}\left|D_{jj}u^\epsilon\right|\\
&\leqslant \left\|\beta_\epsilon(u_\epsilon)\Gamma\left(|\nabla u_\epsilon|^2\right)\right\|_\infty + (n-1)C\\
&\leqslant C,
\end{aligned}
$$

and the proof of $C^{1,1}$ regularity for u is concluded. □

4. Refinement of tools and Lipschitz regularity for high energy activation problem

This section is devoted to prove Theorems 1.7 and 1.8. We shall refine our previously developed tools to handle high energy activation phenomenon. The arguments presented in this section can be adapted to a variety of other problems, where non-degeneracy condition (1.7) in Theorem 1.1 is violated.

Proof of Theorem 1.8. Initially we obtain a gradient control in the "well-behaved" region:

$$\Omega_\varepsilon := \{X \in \Omega' \mid u_\varepsilon \geqslant \lambda_\varepsilon \varepsilon\}.$$

We first note that L^∞ bounds for u_ε uniform in ε follows from classical Alexandroff–Bakelman–Pucci maximum principle. Indeed, in $|u_\varepsilon| \geqslant \varepsilon$, $F(D^2 u_\varepsilon) = 0$, thus

$$|u_\varepsilon| \leqslant \|f\|_{L^\infty(\partial\Omega)}.$$

For the gradient bound in Ω_ε, we just notice that since ζ_ε is supported in $[0, \varepsilon]$,

$$F(D^2 u_\varepsilon) = 0 \quad \text{in } \{u_\varepsilon \geqslant \lambda_\varepsilon \varepsilon\}.$$

Uniform in ε gradient estimate for u_ε follows from standard elliptic regularity theory up to the boundary.

We now turn to the more challenging part: control of ∇u_ε in the transition region

$$\Gamma_\varepsilon := \{y \in \Omega' \mid |u_\varepsilon(y)| \leqslant \lambda_\varepsilon \varepsilon\}.$$

Our strategy shall rely on the analysis of the auxiliary function

$$2v_\varepsilon(x) := |\nabla u_\varepsilon|^2 + \frac{L}{\varepsilon^2} u_\varepsilon^2.$$

Here L is a positive large constant to be chosen a fortiori. We will hereafter drop the subscript ε in u_ε and v_ε. Following our initial path, we compute

$$D_i v = \sum_{k=1}^n D_k u D_{ik} u + L\varepsilon^{-2} u D_i u. \tag{4.1}$$

Differentiating once more we obtain

$$D_{ij} v = \sum_{k=1}^n \{D_{kj} u D_{ik} u + D_k u D_{ij} u_k\} + L\varepsilon^{-2}\{D_j u D_i u + u D_{ij} u\}. \tag{4.2}$$

Let X_0 be a maximum point for v in Γ_ε, that is

$$v(X_0) = \max_{\Gamma_\varepsilon} v. \tag{4.3}$$

From gradient estimate in $\{u_\varepsilon \geqslant \lambda_\varepsilon \varepsilon\}$, we may assume X_0 is an interior point of Ω. If we differentiate the PDE in the kth-direction we obtain

$$\sum_{i,j} F_{ij}(D^2 u) D_{ij} u_k = \frac{1}{\varepsilon^2} \zeta'\left(\frac{u}{\varepsilon}\right) u_k G(x) + \frac{1}{\varepsilon} \zeta\left(\frac{u}{\varepsilon}\right) G'(x). \tag{4.4}$$

By ellipticity, the matrix $A_{ij} := F_{ij}(D^2u(X_0))$ is positive, therefore

$$\begin{aligned} 0 &\geqslant \sum_{i,j} A_{ij} D_{ij} v(X_0) \\ &= \operatorname{Tr}\big(D^2u A_{ij} D^2u\big) + \sum_k u_k A_{ij} D_{ij} u_k + L\varepsilon^{-2}\{A_{ij} D_i u D_j u + u A_{ij} D_{ij} u\} \\ &\geqslant \sum_k u_k A_{ij} D_{ij} u_k + L\varepsilon^{-2}\{A_{ij} D_i u D_j u + u A_{ij} D_{ij} u\}. \end{aligned} \tag{4.5}$$

Above expression is evaluated at X_0 and, as usual, the sum in ij has been omitted. If we multiply (4.12) by u_k and sum it up, we find using Cauchy–Schwarz inequality

$$\begin{aligned} \sum_k u_k \sum_{ij} A_{ij} D_{ij} u_k &= \frac{1}{\varepsilon^2}\zeta'\left(\frac{u}{\varepsilon}\right) G(X_0)|\nabla u|^2 + \frac{1}{\varepsilon}\zeta\left(\frac{u}{\varepsilon}\right) G'(X_0) \sum_k u_k \\ &\geqslant \frac{1}{\varepsilon^2}\zeta'\left(\frac{u}{\varepsilon}\right) G(X_0)|\nabla u|^2 - \sqrt{d}\frac{1}{\varepsilon}\zeta\left(\frac{u}{\varepsilon}\right)\big|G'(X_0)\big||\nabla u|. \end{aligned} \tag{4.6}$$

Returning to (4.13) and using once more ellipticity of A_{ij} we can write,

$$0 \geqslant \varepsilon^{-2}\big(\zeta' G|\nabla u|^2 - \varepsilon\sqrt{d}\zeta\big|G'\big||\nabla u| + \lambda L\big\{|\nabla u|^2 - |u| c_3 \varepsilon^{-1}\zeta G\big\}\big), \tag{4.7}$$

where c_3 depends only upon ellipticity. Organizing inequality (4.7), we reach

$$\big\{\zeta' G + \lambda L\big\}|\nabla u|^2 - \varepsilon\sqrt{d}\zeta\big|G'\big||\nabla u| \leqslant \lambda L c_3 |u| \varepsilon^{-1}\zeta G. \tag{4.8}$$

If we choose

$$L := 2\frac{\max|\zeta'| \cdot \|G\|_{L^\infty}}{\lambda},$$

estimate (4.8) becomes (recall $|u| \leqslant C\varepsilon$)

$$C_4|\nabla u|^2 - C_5\varepsilon|\nabla u| \leqslant C_6|u|\varepsilon^{-1}\zeta(u/\varepsilon) \leqslant CC_6 \tag{4.9}$$

for $C_4 = \max|\zeta'| \cdot \|G\|_{L^\infty}$, $C_5 = \sqrt{d}\|\zeta\|_{L^\infty}\|G'\|_{L^\infty}$ and $C_6 = c_3 \max|\zeta'| \cdot \|G\|^2_{L^\infty}$.

It now follows easily from (4.9) that

$$\big|\nabla u(X_0)\big| \leqslant C_7,$$

where C_7 depends only on dimension, ellipticity, $\|\zeta\|_{C^1}$, $\|G\|_{C^1}$, but is independent of ε. Finally, within the transition area Γ_ε we conclude

$$\big|\nabla u(X)\big|^2 \leqslant v(X) \leqslant C_7 + C^2 \cdot L := C_8,$$

which again is independent of ε. The proof of Theorem 1.8 is completed. □

We now turn our attention to prove Theorem 1.7. As we shall see its proof is a fine combination of ideas from the proof of Theorem 1.8 with the regularizing technics explored in the proof of Theorem 1.5.

Proof of Theorem 1.7. Let β_ε be a $C^2(\mathbb{R})$ non-negative regularized potential satisfying

$$\beta_\varepsilon(s) := \begin{cases} (\alpha+1)s^\alpha & \text{if } s \geqslant \varepsilon, \\ 0 & \text{if } s \leqslant 0. \end{cases}$$

Such a function can be easily constructed exactly as indicated in the proof of Theorem 1.5. As argued in the proof of Theorem 1.5, we can find strong solutions to

$$\begin{cases} F\left(D^2 u_\varepsilon\right) = \beta_\varepsilon(u_\varepsilon) & \text{in } \Omega, \\ u_\varepsilon = f & \text{on } \partial\Omega. \end{cases} \tag{4.10}$$

Our main task now is to establish local $C^{1,\gamma}$ regularity uniform in ε for the family of above solutions u_ε, where $\gamma := \frac{1-|\alpha|}{1+|\alpha|}$. For that, we will analyze the following auxiliary function

$$v_\varepsilon(X) := \psi(u_\varepsilon)|\nabla u_\varepsilon|^2, \quad \text{for } \psi(t) = |t|^\theta,$$

where

$$\theta := -(1+\alpha).$$

Notice that for each $\varepsilon > 0$ fixed, u_ε is strictly positive. Indeed, suppose for sake of contradiction that $\{u_\varepsilon < 0\}$ were non-empty. If so, within such a set u_ε would satisfy a homogeneous elliptic PDE. Furthermore, it is non-negative on $\partial\{u_\varepsilon > 0\}$. Thus, by maximum principle it would have to be strictly positive in that set, which clearly is a contradiction on the definition of the set.

We shall drop the subscript ε in u_ε and v_ε, writing them, hereafter simply as u and v.

Hereafter in the proof, we shall fix a positive C^2 smooth up to the boundary function that vanishes on $\partial\Omega$ and satisfies $|\nabla\varphi|^2 = \mathrm{O}(\varphi)$. Let X_0 be a maximum point for $\omega := \varphi \cdot v$ in Ω, that is

$$\varphi(X_0) \cdot v(X_0) = \max_{\overline{\Omega}} \omega. \tag{4.11}$$

Clearly we may assume X_0 is an interior point. Simple calculus gives

$$D_i\omega = \varphi_i v + \varphi v_i \quad \text{and} \quad D_{ij}\omega = \varphi_{ij} v + \varphi_i v_j + \varphi_j v_i + \varphi v_{ij}.$$

If we differentiate the PDE in the kth-direction we obtain

$$\sum_{i,j} F_{ij}\left(D^2 u\right) D_{ij} u_k = \beta_\varepsilon'(u) u_k. \tag{4.12}$$

By ellipticity, the matrix $A_{ij} := F_{ij}(D^2u(X_0))$ is positive, therefore

$$\begin{aligned} 0 &\geqslant \sum_{i,j} A_{ij} D_{ij}\omega(X_0) \\ &= v\sum_{i,j} A_{ij} D_{ij}\varphi + 2\,\mathrm{Tr}(A_{ij}\nabla\varphi\otimes\nabla v) + \varphi\sum_{i,j} A_{ij} D_{ij} v(X_0). \end{aligned} \tag{4.13}$$

From the fact that X_0 is a critical point of ω, we conclude

$$\nabla v(X_0) = -v\frac{\nabla\varphi(X_0)}{\varphi(X_0)}. \tag{4.14}$$

Ellipticity of A_{ij} and analytic properties of φ,

$$v\sum_{i,j} A_{ij} D_{ij}\varphi + 2\,\mathrm{Tr}(A_{ij}\nabla\varphi\otimes\nabla v) \geqslant -C_0 v, \tag{4.15}$$

where

$$C_0 := 10\Lambda\cdot\max_{\overline{\Omega}}\left\{\left|D^2\varphi\right| + \frac{|\nabla\varphi|^2}{\varphi}\right\}.$$

Let us now focus our attention on the term $\sum_{i,j} A_{ij} D_{ij} v(X_0)$. Simply one verifies that

$$D_i v = \psi'(u)u_i|\nabla u|^2 + 2\psi(u)\sum_k u_k u_{ki}.$$

Differentiating above expression, one reaches

$$\begin{aligned} D_{ij} v = {} & \big(\psi''(u)u_i u_j + \psi'(u)u_{ij}\big)|\nabla u|^2 + 2\psi'(u)u_i\sum_k u_k u_{kj} \\ & + 2\psi'(u)u_j\sum_k u_k u_{ki} + 2\psi(u)\sum_k\{u_{kj}u_{ki} + u_k u_{kij}\}. \end{aligned}$$

It follows from (4.14) that, at X_0,

$$S_j := \sum_k u_k u_{kj} = -\frac{1}{2\psi(u)}\left\{\psi'(u)u_j|\nabla u|^2 + v\frac{\varphi_j}{\varphi}\right\}.$$

Thus

$$\begin{aligned} A_{ij} D_{ij} v = {} & \left[\psi''(u) - 2\frac{(\psi'(u))^2}{\psi(u)}\right] A_{ij}\nabla u\otimes\nabla u|\nabla u|^2 - 2\frac{v}{\varphi}A_{ij}\nabla u\otimes\nabla\varphi \\ & + \psi'(u)A_{ij}u_{ij}|\nabla u|^2 + 2\psi(u)\big(\mathrm{Tr}\big(D^2u A_{ij} D^2u\big) + \beta_\varepsilon'(u)|\nabla u|^2\big). \end{aligned} \tag{4.16}$$

By virtue of ellipticity, the following estimates hold

$$A_{ij}\nabla u\otimes\nabla u\geqslant\lambda|\nabla u|^2, \tag{4.17}$$

$$|A_{ij}\nabla u\otimes\nabla\varphi|\leqslant\Lambda|\nabla u|\cdot|\nabla\varphi|, \tag{4.18}$$

$$|A_{ij}u_{ij}|\leqslant\Lambda F\big(D^2u\big)=\Lambda\beta_\varepsilon(u), \tag{4.19}$$

$$\operatorname{Tr}\big(D^2uA_{ij}D^2u\big)\geqslant 0, \tag{4.20}$$

$$\left[\psi''(u)-2\frac{(\psi'(u))^2}{\psi(u)}\right]=\gamma(\alpha)u^{-(1+\alpha)}\cdot u^{-2}. \tag{4.21}$$

In the last expression above, (4.21), $\gamma(\alpha)=-\alpha-\alpha^2>0$. If we plug these estimates in (4.16), taking into account that $\beta_\varepsilon(u)\leqslant|u|^\alpha$ and $|\beta'_\varepsilon(u)|\leqslant|\alpha||u|^{\alpha-1}$ we find

$$\begin{aligned}A_{ij}D_{ij}v=\gamma(\alpha)vu^{-2}|\nabla u|^2-2\Lambda v\frac{|\nabla\varphi|}{\varphi}|\nabla u|\\ -(1+\alpha)\Lambda u^{-1}|\nabla u|^2-2|\alpha|u^{-2}|\nabla u|^2.\end{aligned} \tag{4.22}$$

In addition, we note that, in the region $|u_\varepsilon|\geqslant 1$, $F(D^2u_\varepsilon)$ is uniformly bounded, independently of ε. Thus, by Alexandroff–Bakelman–Pucci maximum principle,

$$|u_\varepsilon|\leqslant C_1,$$

for a constant C_1 that does not depend upon ε. Combining (4.13), (4.15) and (4.22), and taking into account that $|\nabla\varphi|=O(\sqrt{\varphi})$, we reach

$$C_0u^{-(1+\alpha)}|\nabla u|^2\geqslant u^{-2}|\nabla u|^2\varphi\big(\gamma(\alpha)v-C_2\big)-C_3\sqrt{\varphi}|\nabla u|u^{-(1+\alpha)}|\nabla u|^2.$$

By simple considerations, we can assume $|\nabla u(X_0)|u(X_0)\neq 0$. Thus above estimate becomes

$$C_4\geqslant\gamma(\alpha)\varphi(X_0)\cdot v(X_0)-C_5\sqrt{\varphi(X_0)v(X_0)}.$$

Clearly the above estimate implies that

$$\varphi(X)v(X)\leqslant\varphi(X_0)v(X_0)\leqslant C, \tag{4.23}$$

for a constant C that depends only on dimension, ellipticity, α, $\|f\|_\infty$ and φ, but is independent of ε. In particular, for any subdomain $\Omega'\Subset\Omega$,

$$|u_\varepsilon|^{-(1+\alpha)}(X)\big|\nabla u_\varepsilon(X)\big|^2\leqslant C\big(\Omega'\big),\quad\forall X\in\Omega'. \tag{4.24}$$

Classical considerations now imply that $\|u_\varepsilon\|_{C^{1,\frac{1-|\alpha|}{|\alpha|+1}}}$ is locally bounded, uniformly in ε. By Ascoli–Arzela Theorem, up to a subsequence, u_ε converges uniformly to a locally $C^{1,\frac{1-|\alpha|}{|\alpha|+1}}$ continuous function u. It is now standard to verify that u is a viscosity solution in the weak-star topology sense to

$$F\big(D^2u\big)=(\alpha+1)\big(u^+\big)^\alpha\chi_{\{u>0\}}$$

and that $u^\alpha \in L^1_{\mathrm{loc}}(\{u > 0\})$. We shall omit the details. The proof of Theorem 1.7 is completed. □

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